

# ON THE TYPE IIB SOLUTIONS TO MEAN CURVATURE FLOW

LIANG CHENG

**ABSTRACT.** In this paper we study the Type Iib mean curvature flow for which has the smooth solution exists for all  $t > 0$  and satisfies  $\sup_{M^n \times (0, +\infty)} t|A|^2 = \infty$ , where  $A(\cdot, t)$  is the second fundamental form. We prove that the long-time solution to mean curvature flow for entire graphs satisfying certain conditions could be Type Iib. Our results lead us to getting the nontrivial examples for the Type Iib mean curvature flow. In order to prove the main theorems, we extend Andrews' noncollapsing theorem to noncompact case. As the application, we also show that the limit of suitable rescaling sequence for mean-convex Type Iib mean curvature flow satisfying  $\delta$ -Andrews' noncollapsing condition is translating soliton.

## 1. INTRODUCTION

Let  $x_0 : M^n \rightarrow \mathbb{R}^{n+1}$  be a complete immersed hypersurface. Consider the mean curvature flow

$$\frac{\partial x}{\partial t} = \vec{H}, \quad (1.1)$$

with the initial data  $x_0$ , where  $\vec{H} = -H\nu$  is the mean curvature vector and  $\nu$  is the outer unit normal vector. Denote the images  $x(M^n, t) = M_t$ . The mean curvature flow always blows up at finite time on closed hypersurfaces. However, the mean curvature flow for noncompact hypersurfaces may have a smooth solution which exists for all time  $t > 0$ , for which we call it the long-time solution. Ecker and Huisken [9] showed that the mean curvature flow on locally Lipschitz continuous entire graph over  $\mathbb{R}^n$  has a long-time solution. Notice that if  $M_0$  can be written as an entire graph, then the parabolic system (1.1) up to tangential diffeomorphisms is equivalent to the following quasilinear equation

$$\frac{du}{dt} = \sqrt{1 + |Du|^2} \operatorname{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) \quad (1.2)$$

---

2000 *Mathematics Subject Classification.* Primary 53C44; Secondary 53C42, 57M50.

*Key words and phrases.* Type Iib mean curvature flows; Entire graphs; Andrews' noncollapsing theorem for noncompact hypersurfaces.

Liang Cheng's Research partially supported by the Natural Science Foundation of China 11201164.

(see [9]), where  $u$  is the graph representation for  $M_t$ .

Analogous to which was introduced by Hamilton [10] for Ricci flow, one can classify the long-time solutions to the mean curvature flow into the following two types :

**Definition 1.1.** The long-time solution to the mean curvature flow  $M_t$  is called

(1) Type IIb if  $\sup_{M^n \times (0, +\infty)} t|A|^2 = \infty$ ,

(2) Type III if  $\sup_{M^n \times (0, +\infty)} t|A|^2 < \infty$ ,

where  $A(\cdot, t)$  is the second fundamental form of  $M_t$ .

In this paper we study the Type IIb mean curvature flow. The nontrivial examples for the Type IIb mean curvature flow are gotten. We also study the singularity formation for the Type IIb mean curvature flow.

Recall the singularity formation of the mean curvature flow on closed hypersurfaces at the first singular time was described by Huisken [10] as follows: The solution to mean curvature flow  $M_t$  on closed hypersurfaces which blows up at first finite time  $T$  is called

(1) Type I if  $\sup_{M^n \times [0, T)} (T - t)|A|^2 < \infty$ ,

(2) Type II if  $\sup_{M^n \times [0, T)} (T - t)|A|^2 = \infty$ .

Using a monotonicity formula, Huisken [10] showed that Type I singularities of mean curvature flow are smooth asymptotically like self-shrinkers. If one choose an essential blowup sequence  $(p_j, t_j)$  for the compact Type II mean curvature flow such that  $t_j \in [0, T - \frac{1}{j}]$ ,  $p_j \in M^n$ , and

$$H^2(p_j, t_j)(T - \frac{1}{j} - t_j) = \max_{M^n \times [0, T - \frac{1}{j}]} H^2(p, t)(T - \frac{1}{j} - t) \quad (1.3)$$

Let  $L_j = |H(p_j, t_j)|$ . Consider the following rescaled mean curvature flows

$$M_t^j = L_j(M_{t_j + L_j^{-2}t} - x(p_j, t_j)), \quad (1.4)$$

for  $t \in [\alpha_j, \Omega_j]$ , where  $\alpha_j = -t_j L_j^2 \rightarrow -\infty$  and  $\Omega_j = (T - t_j - \frac{1}{j})L_j^2 \rightarrow +\infty$ .

For each rescaled flow (1.4),  $0 \in M_0^j$  and  $|H_j|$  achieves the maximum value 1 at  $t = 0$ . By employing a Harnack inequality, Hamilton [14] showed that any strictly convex eternal solution to the mean curvature flow where the mean curvature assumes its maximum value at a point in space-time must be a translating soliton. Huisken and Sinestrari ([11] [12]) proved blowup sequence of the compact mean curvature flow with positive mean curvature subconverges to a weakly convex limit splitting as  $\mathbb{R}^{n-k} \times \Sigma^k$ , where  $\Sigma^k$  is strictly convex. Their results implies that rescaled sequence (1.4) subconverges to a translating soliton if  $M_0$  is mean-convex.

Similar to the Type II mean curvature flow, if  $M_t$  is the Type IIB mean curvature flow with bounded second fundamental form at each time slice, as which was introduced by Hamilton [15] for Ricci flow, one can choose  $j \rightarrow +\infty$ , and pick  $P_j$  and  $t_j$  such that

$$t_j(j - t_j)H^2(P_j, t_j) \geq \gamma_j \sup_{M^n \times [0, j]} t(j - t)H^2(P, t), \quad (1.5)$$

where  $\gamma_j \nearrow 1$ . Let  $L_j = |H|(P_j, t_j)$ . Consider the following the rescaled mean curvature flows

$$M_t^j = L_j(M_{t_j + L_j^{-2}t} - x(P_j, t_j)), \quad (1.6)$$

for  $t \in [\alpha_j, \Omega_j]$ , where  $\alpha_j = -t_j L_j^2$  and  $\Omega_j = (j - t_j)L_j^2$ . Then

$$H_j^2(\cdot, t) \leq \gamma_j^{-1} \frac{\alpha_j}{\alpha_j - t} \frac{\Omega_j}{\Omega_j - t},$$

for  $t \in [\alpha_j, \Omega_j]$ . Now

$$\frac{1}{-\alpha_j^{-1} + \Omega_j^{-1}} \geq \gamma_j j^{-1} \sup_{M^n \times [0, j]} (t(j - t)H^2(x, t)) \geq \frac{\gamma_j}{2} \sup_{M^n \times [0, \frac{j}{2}]} tH^2(x, t) \rightarrow +\infty,$$

Hence  $\alpha_j \rightarrow -\infty$  and  $\Omega_j \rightarrow +\infty$ . If the Type IIB mean curvature flow is convex, one get the limit  $M_\infty$  of the rescaled sequence (1.6) is an eternal solution splitting as  $\mathbb{R}^{n-k} \times \Sigma^k$  with its mean curvature achieves the maximum value 1 in the space-time, where  $\Sigma^k$  is strictly convex. This implies that  $M_\infty$  is a translating soliton by Hamilton's Harnack inequality. Moreover, if the Type IIB mean curvature flow is mean-convex and satisfies the  $\delta$ -Andrews' noncollapsing condition at  $t = 0$ , we can also get the limit  $M_\infty$  of the rescaled sequence (1.6) is weakly convex (see Theorem 1.8). Hence  $M_\infty$  is still a translating soliton by Hamilton's Harnack inequality.

For the Type III mean curvature flow, rescaling the mean curvature flow as

$$\tilde{x}(\cdot, s) = \frac{1}{\sqrt{2t + 1}} x(\cdot, t), \quad (1.7)$$

where  $s$  is given by  $s = \frac{1}{2} \log(2t + 1)$ . The normalized mean curvature flow then becomes

$$\frac{\partial \tilde{x}}{\partial s} = \vec{\mathbf{H}} - \tilde{x}. \quad (1.8)$$

Note that Type III condition implies  $\sup_{M^n \times (0, +\infty)} |\tilde{A}| < \infty$ . If the Type III mean curvature flow is convex, one can use Hamilton's Harnack inequality to get the limit  $M_\infty$  is a non-flat self-expander splitting as  $\mathbb{R}^{n-k} \times \Sigma^k$ , where  $\Sigma^k$  is strictly convex (see Corollary 6.3 in the appendix). We remark that a counter-example in [5](see Example 3.4 in [5]) shows that the rescaled sequence (1.7) can not converge to the self-expander if only assuming the

Type III mean curvature flow is mean-convex. Recently the author and Sesum [5] also introduced monotonicity formulas related to self-expanders and showed the normalized flow (1.8) for Type III mean curvature flow subconverges to the self-expander under certain conditions.

Typical examples of the Type III mean curvature flow are evolving entire graphs satisfying the following condition

$$\nu := \langle \nu, w \rangle^{-1} \leq c, \quad (1.9)$$

which in particular implies the entire graphs having the bounded gradient, where  $\nu$  is the unit normal vector of the graph and  $w$  is a fixed unit vector such that  $\langle \nu, w \rangle > 0$ . Ecker and Huisken showed that the mean curvature flow on entire graphs satisfying the condition (1.9) is Type III (Corollary 4.4 in [8]). Moreover, Ecker and Huisken [9] also proved that if the entire graph satisfies condition (1.9) and the estimate

$$\langle x_0, \nu \rangle^2 \leq c(1 + |x_0|^2)^{1-\delta} \quad (1.10)$$

at time  $t = 0$ , where  $c < \infty$  and  $\delta > 0$ , then the solution to the normalized mean curvature flow (1.8) with initial data  $x_0$  converges as  $s \rightarrow \infty$  to a self-expander.

In contrast to Type III mean curvature flow, much less examples are known about the Type IIb mean curvature flow except non-flat translating solitons. In this paper we prove that the mean curvature flow for entire graphs satisfying certain conditions can be Type IIb. Our results lead us to getting the nontrivial examples for the Type IIb mean curvature flow.

The first theorem of the paper is following

**Theorem 1.2.** *Let  $M_t$  be a solution to the mean curvature flow for immersed noncompact hypersurface in  $\mathbb{R}^{n+1}$ . Suppose that there exist a fixed vector  $\omega$  and constants  $C_1, C_2$  such that*

$$C_1 H \leq W \leq C_2 H, \quad (1.11)$$

*at  $t = 0$ , where  $W = \langle \nu, \omega \rangle$ , and*

$$M_0 \text{ can be contained in the half-plane } \mathbb{R}_+^{n+1} \text{ with its boundary } \partial \mathbb{R}_+^{n+1} \text{ not parallel to } \omega. \quad (1.12)$$

*Then the mean curvature flow  $M_t$  can not be Type III. In addition, if  $M_0$  is an entire graph, then the long-time solution to mean curvature flow for  $M_0$  must be Type IIb.*

**Remark 1.3.** Let  $M_0$  be the hyperplane in  $\mathbb{R}^{n+1}$ . We choose  $\omega$  be a fixed vector which is parallel to the hyperplane, and hence  $H = W \equiv 0$ . Then  $M_0$  satisfies the condition (1.11) rather than condition (1.12), and clearly the mean curvature flow with initial data  $M_0$  is not Type IIb. The example shows that the condition (1.12) in Theorem 1.2 can not be removed.

Let  $M_0 = (y, |y|^\alpha)$  be the entire graph over  $\mathbb{R}^n$  with  $n \geq 2$ . We can use Theorem 1.2 to show that the mean curvature flow with initial data  $M_0$  for the case  $\alpha \geq 2$  must be Type Iib (see Corollary 2.3). Notice that such graph for the case  $0 < \alpha \leq 1$  satisfying the linear growth condition (1.9), so the mean curvature flow for such graph is Type III by Ecker and Huisken's result [8]. It remains to determine the Type of the mean curvature flow for such graph for the case  $1 < \alpha < 2$ .

The next theorem of this paper implies the mean curvature flow with initial data  $M_0$  for the case  $\alpha > 1$  must be Type Iib (see Corollary 4.1). Before presenting the second theorem of this paper, we recall the definition of  $\delta$ -Andrews' noncollapsing condition.

**Definition 1.4.** [1][6][7] ( $\delta$ -Andrews' noncollapsing condition) If  $M$  is a smooth, complete, mean-convex embedded hypersurface (possibly noncompact) with  $M = \partial K$ , then  $M$  satisfies the  $\delta$ -Andrews' noncollapsing condition for  $\delta > 0$  if for every  $p \in M$  there are closed balls  $\bar{B}_{Int} \subseteq K$  and  $\bar{B}_{Ext} \subseteq \mathbb{R}^{n+1} \setminus Int(K)$  of radius at least  $\frac{\delta}{H(p)}$  that are tangent to  $M$  at  $p$  from the interior and exterior of  $M$  respectively.

The second theorem of this paper is the following

**Theorem 1.5.** *Let  $M_t$ ,  $t \in [0, T)$ , be the solution to the mean-convex mean curvature flow for the embedded complete noncompact hypersurface in  $\mathbb{R}^{n+1}$ . Suppose that there exist a fixed vector  $\omega$  and positive constants  $\epsilon$ ,  $C$  such that*

$$|W| \leq C(1 + |x|^2)^{\frac{1-\epsilon}{2}} H, \quad (1.13)$$

*at  $t = 0$ , where  $W = \langle v, \omega \rangle$ , and  $M_0$  satisfies the  $\delta$ -Andrews' noncollapsing condition and can be contained in the half-plane  $\mathbb{R}_+^{n+1}$  with its boundary  $\partial\mathbb{R}_+^{n+1}$  not parallel to  $\omega$ . Then the mean curvature flow  $M_t$  can not be Type III. In addition, if  $M_0$  is an entire graph, then the long-time solution to mean curvature flow for  $M_0$  must be Type Iib.*

**Remark 1.6.** (1) Since  $\delta$ -Andrews' noncollapsing condition is needed in Theorem 1.5, Theorem 1.5 can not cover Theorem 1.2. For example, the grim reaper satisfies the conditions of Theorem 1.2 rather than the conditions of Theorem 1.5.

(2) A key step in the proof of Theorem 1.5 is to prove if the noncompact hypersurface  $M_0$  satisfies the  $\delta$ -Andrews' noncollapsing condition, then the following estimate

$$|\nabla^l A| \leq CH^{l+1}, \quad (1.14)$$

holds under the mean curvature flow for all  $t > 0$  and any integer  $l$  (Corollary 3.3). The estimate (1.14) was obtained by Huisken and Sinestrari [13] for compact mean-convex mean curvature flow. Then Haslhofer and Kleiner

[7] gave a another proof for the estimate (1.14) for the compact case by using Andrews' noncollapsing theorem [1]. In order to prove the estimate (1.14) we extend Andrews' noncollapsing theorem to the noncompact case. Precisely, we get the following

*Theorem 1.7. Let  $M_t$  be a solution to the mean curvature flow for mean-convex complete noncompact embedded hypersurface in  $\mathbb{R}^{n+1}$  with bounded second fundamental form at each time slice. If  $M_0$  satisfies the  $\delta$ -Andrews' noncollapsing condition, then it remains so under the mean curvature flow.*

Then a local estimate by Haslhofer and Kleiner [7] (see Corollary 2.15 in [7]) implies the estimate (1.14). Another application to Theorem 1.7 is that we can show that

*Theorem 1.8. If the Type IIb mean curvature flow is mean-convex and satisfies the  $\delta$ -Andrews' noncollapsing condition at  $t = 0$ , then limit  $M_\infty$  of the rescaled sequence (1.6) is a weakly convex and hence  $M_\infty$  is the translating soliton splitting as  $\mathbb{R}^{n-k} \times \Sigma^k$ , where  $\Sigma^k$  is strictly convex.*

Finally, by applying Ecker and Huisken's result in [9] to the case  $0 < \alpha \leq 1$  (smooth out the graph at the origin) and summarizing the results in this paper about the mean curvature flow for the entire graph  $(y, |y|^\alpha)$  over  $\mathbb{R}^n$ ,  $n \geq 2$ , we have the following table.

TABLE 1. Mean curvature flow for graph  $(y, |y|^\alpha)$  over  $\mathbb{R}^n$ ,  $n \geq 2$  (smooth out the graph at the origin when  $0 < \alpha < 1$ )

	When $t \rightarrow +\infty$ , for any $p \in M^n$	Singularity Type	Asymptotic behavior
$0 < \alpha \leq 1$	$ A (p, t) \rightarrow 0$	Type III	rescaled sequence (1.7) subconverges to a self-expander
$1 < \alpha < 2$	$ A (p, t) \rightarrow 0$	Type IIb	rescaled sequence (1.6) subconverges to a translating soliton
$\alpha = 2$	$0 < c_p \leq  A (p, t) \leq 2n^2$		
$\alpha > 2$	$ A (p, t) \rightarrow +\infty$		

*Remark 1.9.* Table 1 is also true for the more generalized case for the mean curvature flow of the convex entire graph  $(y, u(|y|))$  over  $\mathbb{R}^n$  with  $\lim_{r \rightarrow +\infty} \frac{u(r)}{r^\alpha} = c > 0$ ,  $n \geq 2$ . All the proofs are similar, we omit the details here.

The structure of this paper is as follows. In section 2 we give proofs of Theorem 1.2 and its application Corollary 2.3. In section 3, we extend Andrews' noncollapsing theorem to noncompact case (Theorem 1.7) and give the proofs of estimate (1.14) and Theorem 1.8. In section 4 we give

proofs of Theorem 1.5 and its application Corollary 4.1. In section 5 we give the proof of Table 1.

## 2. PROOF OF THEOREM 1.2 AND ITS APPLICATION

Before presenting the proof of Theorem 1.2 we need prove the following two lemmas.

**Lemma 2.1.** *Define  $W = \langle \nu, \omega \rangle$ , where  $\omega$  is a fixed vector. Under the mean curvature flow, we have*

$$\left(\frac{\partial}{\partial t} - \Delta\right)W = |A|^2 W.$$

*Proof.* By Lemma 3.3 in [10], we have

$$\frac{\partial}{\partial t} \nu = \nabla H. \quad (2.1)$$

Moreover, we compute at some point with normal coordinates

$$\Delta W = e_i \langle \nabla_{e_i} \nu, \omega \rangle = e_i \langle h_{il} e_l, \omega \rangle = \langle \nabla H, \omega \rangle - |A|^2 W. \quad (2.2)$$

Then Theorem 2.1 follows from (2.1) and (2.2).  $\square$

**Lemma 2.2.** *Let  $M_t$  be a solution to the mean curvature flow with bounded second fundamental form at each time slice. Define  $W = \langle \nu, \omega \rangle$ , where  $\omega$  is some fixed vector. If  $H \geq CW$  or  $H \leq CW$  for some constant  $C$  at  $t = 0$ , then it remains so under the mean curvature flow.*

*Proof.* By Lemma 2.1, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)(H - CW) = |A|^2(H - CW).$$

By the maximum principle (Theorem 4.3 in [9]),  $H - CW \geq 0$  or  $H - CW \leq 0$  is preserved under the mean curvature flow.  $\square$

Now we give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** We argue by contradiction. Assume that  $M_t = x(M^n, t)$  is the Type III solution to the mean curvature flow with  $\sup_{M^n \times [0, \infty)} t|A|^2 =$

$C < \infty$  and  $\tilde{x}(\cdot, s)$  be its corresponding normalized mean curvature flow (1.8). Denote  $\tilde{M}_s = \tilde{x}(M^n, s)$ . It follows from Lemma 2.2 that  $C_1 H \leq W \leq C_2 H$  holds for all  $t \geq 0$ . By (1.7), we have  $\sup_{M^n \times [0, \infty)} |\tilde{A}|^2 = C < \infty$ ,  $\tilde{W} = W$

and  $\tilde{H} = \sqrt{2t+1}H$ . Hence

$$-\frac{nCC'}{\sqrt{2t+1}} \leq |\tilde{W}| \leq \frac{nCC'}{\sqrt{2t+1}}, \quad (2.3)$$

for  $C' = \max |C_1|, |C_2|$ , which implies that

$$|\tilde{W}| \rightarrow 0 \quad (2.4)$$

as  $s \rightarrow +\infty$ .

We calculate that

$$\frac{\partial}{\partial s} |\tilde{x}|^2 = 2\langle \vec{\mathbf{H}}, \tilde{x} \rangle - 2|\tilde{x}|^2. \quad (2.5)$$

It follows that

$$|\tilde{x}|(p, s) \leq e^{-s} |\tilde{x}_0|(p) + nC(1 - e^{-s}),$$

for any  $p \in M^n$ . Hence for any fixed point  $p \in M^n$ ,

$$|\tilde{x}|(p, s) \leq nC + 1 \quad (2.6)$$

for  $s$  sufficiently large.

Notice that

$$M_t \cap B(o, R) \leq C(R, T) \quad (2.7)$$

holds for the mean curvature flow which exists for finite time  $[0, T)$  (see [10]), where  $B(o, R)$  is extrinsic ball in  $\mathbb{R}^n$ . Because lack of (2.7) for the long-time solutions to mean curvature flow, we use the technique in [4] to get an intrinsic limit. Since  $|\tilde{A}| \leq \sqrt{C}$  for  $[0, +\infty)$ , the injectivity radius of  $(M^n, g(s))$  at  $p$  has the positive lower bound only depending on  $C$ . We also have  $|\tilde{\nabla}^m \tilde{A}| \leq C_m$  on  $[0, +\infty)$  by the standard estimates by Ecker and Huisken [9]. It follows that there exists a sequence  $s_i \rightarrow +\infty$  such that  $(M^n, g(s_i), p)$  converges to a complete manifold  $(M_\infty^n, g_\infty, p_\infty)$  in  $C^\infty$  pointed Gromov-Hausdorff sense. That is, for any  $r > 0$  and  $i$ , there exist embeddings  $\phi_i : B_{g_\infty}(p_\infty, r) \rightarrow M^n$  such that  $\phi_i(p_\infty) = p$  and  $\phi_i^* g_i$  converges smoothly to  $g_\infty$  on  $B_{g_\infty}(p_\infty, r)$ , where  $B_{g_\infty}(p_\infty, r)$  is the intrinsic ball on  $(M_\infty^n, g_\infty, p_\infty)$ . Since

$$(\phi_i^* g_i)_{kl} = \partial_k \tilde{x}_i \circ \phi_i \cdot \partial_l \tilde{x}_i \circ \phi_i, \quad (2.8)$$

the first derivatives of  $\tilde{x}_i \circ \phi_i$  are uniformly bounded on  $B_{g_\infty}(p_\infty, r)$ . Moreover, by the Gauss-Weingarten relations and  $|\tilde{\nabla}^m \tilde{A}| \leq C_m$ ,

$$\begin{aligned} \partial_k \partial_l F &= \Gamma_{kl}^q \partial_q F - h_{kl} \nu, \\ \partial_k \nu &= h_{kl} g^{lq} \partial_q F, \end{aligned}$$

here  $F = \tilde{x}_i \circ \phi_i$  and  $g = \phi_i^* g_i$  in our case, we have all derivatives of  $\tilde{x}_i \circ \phi_i$  are uniformly bounded on  $B_{g_\infty}(p_\infty, r)$ . Notice that  $\tilde{x}_i \circ \phi_i : B_{g_\infty}(p_\infty, r) \rightarrow \mathbb{R}^{n+1}$  and  $\tilde{x}_i \circ \phi_i$  is uniformly bounded at  $p_\infty$  by (2.6). It follows that  $\tilde{x}_i \circ \phi_i$  subconverges smoothly to map  $\tilde{x}_\infty : B_{g_\infty}(p_\infty, r) \rightarrow \mathbb{R}^{n+1}$ . Let  $i \rightarrow \infty$  in (2.8), we get  $(g_\infty)_{kl} = \partial_k \tilde{x}_\infty \cdot \partial_l \tilde{x}_\infty$ . Hence  $\tilde{x}_\infty$  is an immersion on  $B_{g_\infty}(p_\infty, r)$ . By the standard diagonal argument and taking  $r = r_l \rightarrow \infty$ , we get the immersion  $\tilde{x}_\infty : M_\infty^n \rightarrow \mathbb{R}^{n+1}$ .



By (2.4) we conclude that  $\tilde{W}_\infty = 0$ , which implies  $\tilde{x}_\infty$  is complete cylindrical hypersurface, i.e. the hypersurface splitting as  $\Sigma^{n-1} \times l$  with  $l$  is a straight line parallel to  $\omega$ . Hence  $\tilde{x}_\infty(M_\infty^n)$  must cross over the plane  $x_{n+1} = 0$ . However the plane  $x_{n+1} = 0$  is steady under the normalized mean curvature flow (1.7), which implies  $\tilde{M}_s$  is contained in  $\mathbb{R}_+^{n+1}$  for all  $s \geq 0$ . It follows that  $\tilde{x}_\infty(M_\infty^n)$  is contained in  $\mathbb{R}_+^{n+1}$ . Then we get a contradiction.  $\square$

As an application to Theorem 1.2, we have following corollary.

**Corollary 2.3.** *Let  $(y, u(|y|))$  be the smooth convex entire graph over  $\mathbb{R}^n$  for  $n \geq 2$ . Suppose that there exist positive constants  $c$  and  $N$  such that for  $r \geq N$*

$$u'(r) \geq cr. \quad (2.9)$$

*Then the long-time solution to mean curvature flow with initial data  $(y, u(|y|))$  must be Type Iib. In particular the long-time solution to mean curvature flow for graph  $(y, |y|^\alpha)$  over  $\mathbb{R}^n$ ,  $n \geq 2$  and  $\alpha \geq 2$ , must be Type Iib.*

*Proof.* We choose  $\omega = -e_{n+1}$ . Define  $r = |y|$ . By direct calculation,

$$W = \frac{1}{(1 + u'(r)^2)^{\frac{1}{2}}} \quad (2.10)$$

and

$$H = \frac{u''(r)}{(1 + u'(r)^2)^{\frac{3}{2}}} + \frac{(n-1)u'(r)}{r(1 + u'(r)^2)^{\frac{1}{2}}}. \quad (2.11)$$

Since  $(y, u(|y|))$  is convex, we have  $u''(r) > 0$  and  $u'(r) > 0$ . Then

$$\begin{aligned} \frac{H}{W} &\geq \frac{(n-1)u'(r)}{r} \\ &\geq c(n-1) \end{aligned}$$

for  $r \geq N$ . It follows that  $0 \leq \frac{W}{H} \leq \frac{1}{c(n-1)}$  for  $r \geq N$ . Since  $W$  and  $H$  are both positive and continuous, (1.11) is satisfied for  $r \leq N$ . Then Corollary 2.3 follows from Theorem 1.2 directly.  $\square$

### 3. ANDREWS' NONCOLLAPSING FOR MEAN-CONVEX NONCOMPACT HYPERSURFACES

In [1] Andrews gave a short quantitative argument about the result proved by Sheng and Wang [17] that the compact mean-convex mean curvature flow satisfies  $\delta$ -Andrews' noncollapsing condition for all time for some  $\delta > 0$ . In this section we extend Andrews' arguments to noncompact case. We first need the following standard barrier function.

**Lemma 3.1** (Lemma 12.30 in [2]). *Let  $(M^n, g)$  be a complete noncompact Riemannian manifold with bounded sectional curvature  $|Rm| \leq k_0$  for some*

$k_0 \geq 0$ . Then there exists constant  $D = D(n, k_0) > 0$  such that for any  $O \in M^n$  there exists a  $C^\infty$  function  $h : M^n \rightarrow \mathbb{R}$  satisfying

$$D^{-1}(d_g(O, x) + 1) \leq h(x) \leq D(d_g(O, x) + 1)$$

and

$$|\nabla_g h| \leq D, \quad \nabla_g \nabla_g h \leq D,$$

on  $M^n$ .

Next we give the proof of Theorem 1.7.

**Proof of Theorem 1.7.** We follow the Andrews' calculation in [1]. Denote

$$Z(x, y, t) = \frac{H(x)}{2} \|X(y, t) - X(x, t)\|^2 + \delta \langle X(y, t) - X(x, t), \nu(x) \rangle. \quad (3.1)$$

We will prove that if  $Z \geq 0$  at  $t = 0$  then  $Z \geq 0$  for all  $t \geq 0$  for any constant  $\delta$ , as showed in [1], that implies Theorem 1.7. Write  $d = d(x, y, t) = \|X(y, t) - X(x, t)\|^2$ ,  $\eta(x, y, t) = \frac{X(y, t) - X(x, t)}{d}$ . We choose the normal coordinates at  $x$  and  $y$ . By the equations (1) and (2) in [1],

$$\frac{\partial Z}{\partial x^i} = -dH_x \langle \eta, \partial_i^x \rangle + \frac{d^2}{2} \nabla_i H_x + \delta d h_{iq}^x g_x^{qp} \langle \eta, \partial_p^x \rangle, \quad (3.2)$$

$$\frac{\partial Z}{\partial y^i} = dH_x \langle \eta, \partial_i^y \rangle + \delta \langle \partial_i^y, \nu_x \rangle. \quad (3.3)$$

Choose local coordinates so that  $\{\partial_i^x\}$  are orthonormal and  $\{\partial_i^y\}$  are orthonormal, and  $\partial_i^x = \partial_i^y$  for  $i = 1, \dots, n-1$ . Thus  $\partial_n^x$  and  $\partial_n^y$  are coplanar with  $\nu_x$  and  $\nu_y$ . Hence  $\partial_n^x - \langle \partial_n^x, \partial_n^y \rangle \partial_n^y = \langle \partial_n^x, \nu_y \rangle \nu_y$ . By the calculation in [1]

$$\begin{aligned} \frac{\partial Z}{\partial t} &= \sum_{i,j=1}^n (g_x^{ij} \frac{\partial^2 Z}{\partial x^i \partial x^j} + 2g_x^{ik} g_y^{jl} \langle \partial_k^x, \partial_l^y \rangle \frac{\partial^2 Z}{\partial x^i \partial y^j} + g_y^{ij} \frac{\partial^2 Z}{\partial y^i \partial y^j}) \\ &= |A^x|^2 Z + 2d \langle \eta, \partial_i^x - \langle \partial_i^x, \partial_k^y \rangle g_y^{kl} \partial_l^y \rangle g_x^{ij} \nabla_j H_x - 2(H_x - \delta h_{nn}^x)(1 - \langle \partial_n^x, \partial_n^y \rangle^2) \\ &= |A^x|^2 Z + 2d \langle \eta, \partial_i^x - \langle \partial_i^x, \partial_k^y \rangle g_y^{kl} \partial_l^y \rangle g_x^{ij} \nabla_j H_x - 2(H_x - \delta h_{nn}^x) \langle \partial_n^x, \nu_y \rangle^2 \end{aligned} \quad (3.4)$$

We get from (3.2) that

$$\begin{aligned} &2d \langle \eta, \partial_i^x - \langle \partial_i^x, \partial_k^y \rangle g_y^{kl} \partial_l^y \rangle g_x^{ij} \nabla_j H_x \\ &= 2d \langle \eta, \partial_n^x - \langle \partial_n^x, \partial_n^y \rangle \partial_n^y \rangle \left( \frac{2}{d} \langle \eta, H_x \partial_n^x - \delta h_{nn}^x \partial_n^x \rangle + \frac{2}{d^2} \frac{\partial Z}{\partial x^n} \right) \\ &= 4(H_x - \delta h_{nn}^x) \langle \eta, \nu_y \rangle \langle \partial_n^x, \nu_y \rangle \langle \eta, \partial_n^x \rangle + \frac{4}{d} \frac{\partial Z}{\partial x^n} \langle \eta, \nu_y \rangle \langle \partial_n^x, \nu_y \rangle. \end{aligned} \quad (3.5)$$

Recall the Lemma 4 in [1]

$$\nu_y \sqrt{1 + \frac{2H_x}{\delta^2} Z - \frac{1}{\delta^2} |\nabla_y Z|^2} = \nu_x + \frac{dH_x}{\delta} \eta - \frac{1}{\delta} \frac{\partial Z}{\partial y^q} g_y^{qp} \partial_p^y, \quad (3.6)$$

by writing  $\rho' = \sqrt{1 + \frac{2H_x}{\delta^2}Z - \frac{1}{\delta^2}|\nabla_y Z|^2}$ , we have

$$\rho' v_y = v_x + \frac{dH_x}{\delta} \eta - \frac{1}{\delta} \frac{\partial Z}{\partial y^q} \partial_q^y. \quad (3.7)$$

By (3.7), we obtain that

$$\begin{aligned} \langle \eta, \partial_n^x \rangle &= \frac{\delta}{dH_x} \langle \rho' v_y - v_x + \frac{1}{\delta} \frac{\partial Z}{\partial y^q} \partial_q^y, \partial_n^x \rangle \\ &= \frac{\delta}{dH_x} \rho' \langle v_y, \partial_n^x \rangle + \frac{1}{dH_x} \frac{\partial Z}{\partial y^q} \langle \partial_q^y, \partial_n^x \rangle. \end{aligned} \quad (3.8)$$

It follows that

$$\begin{aligned} \langle \eta, v_y \rangle \langle \partial_n^x, v_y \rangle \langle \eta, \partial_n^x \rangle &= \frac{\delta}{dH_x} \langle \eta, \rho' v_y \rangle \langle \partial_n^x, v_y \rangle^2 + \frac{1}{dH_x} \frac{\partial Z}{\partial y^q} \langle \eta, v_y \rangle \langle \partial_n^x, v_y \rangle \langle \partial_q^y, \partial_n^x \rangle \\ &= \frac{\delta}{dH_x} \langle \eta, v_x + \frac{dH_x}{\delta} \eta - \frac{1}{\delta} \frac{\partial Z}{\partial y^q} \partial_q^y \rangle \langle \partial_n^x, v_y \rangle^2 \\ &\quad + \frac{1}{dH_x} \frac{\partial Z}{\partial y^q} \langle \eta, v_y \rangle \langle \partial_n^x, v_y \rangle \langle \partial_q^y, \partial_n^x \rangle \\ &= \left( \frac{Z}{d^2 H_x} + \frac{1}{2} \right) \langle \partial_n^x, v_y \rangle^2 - \frac{1}{dH_x} \frac{\partial Z}{\partial y^q} \langle \eta, \partial_q^y \rangle \langle \partial_n^x, v_y \rangle^2 \\ &\quad + \frac{1}{dH_x} \frac{\partial Z}{\partial y^q} \langle \eta, v_y \rangle \langle \partial_n^x, v_y \rangle \langle \partial_q^y, \partial_n^x \rangle, \end{aligned} \quad (3.9)$$

where we use (3.7) in the second equality and  $Z = \frac{H_x}{2} d^2 + \delta d \langle \eta, v_x \rangle$  in the last equality. Combining with (3.4), (3.5), (3.8) and (3.9), we get

$$\begin{aligned} \frac{\partial Z}{\partial t} - \sum_{i,j=1}^n (g_x^{ij} \frac{\partial^2 Z}{\partial x^i \partial x^j} + 2g_x^{ik} g_y^{jl} \langle \partial_k^x, \partial_l^y \rangle \frac{\partial^2 Z}{\partial x^i \partial y^j} + g_y^{ij} \frac{\partial^2 Z}{\partial y^i \partial y^j}) \\ = (|A^x|^2 + \frac{4(H_x - \delta h_{nn}^x)}{d^2 H_x} \langle \partial_n^x, v_y \rangle^2) Z + \frac{4}{d} \frac{\partial Z}{\partial x^i} \langle \eta, v_y \rangle \langle \partial_n^x, v_y \rangle \\ - \frac{4(H_x - \delta h_{nn}^x)}{dH_x} \frac{\partial Z}{\partial y^q} \langle \eta, \partial_q^y \rangle \langle \partial_n^x, v_y \rangle^2 \\ + \frac{4(H_x - \delta h_{nn}^x)}{dH_x} \frac{\partial Z}{\partial y^q} \langle \eta, v_y \rangle \langle \partial_n^x, v_y \rangle \langle \partial_q^y, \partial_n^x \rangle. \end{aligned} \quad (3.10)$$

Assume that second fundamental form for  $M_t$  is bounded by  $C_0$  on  $[0, T]$ . Then  $|\nabla^m A| \leq C_m$  on  $[0, T]$  by the standard estimates by Ecker and Huisken [9]. By Lemma 3.1 and direct calculations, we get for  $h$  which is the function defined in Lemma 3.1

$$D'^{-1}(d_{g(t)}(O, x) + 1) \leq h(x) \leq D'(d_{g(t)}(O, x) + 1)$$

$$|\nabla h| \leq D', \quad \nabla \nabla h \leq D',$$

for  $t \in [0, T]$ , where  $\nabla = \nabla_{g(t)}$  and  $D'$  is a constant only depending on  $D$ ,  $C_0$  and  $C_m$ . Let  $Q(x, y, t) = Z(x, y, t) + \epsilon \zeta(x, y, t)$  where  $\zeta(x, y, t) = \xi(x, t) + \xi(y, t)$  with  $\xi(x, t) = e^{(B+nD'+D'^2)t+h(x)}$  and  $\xi(y, t) = e^{(B+nD'+D'^2)t+h(y)}$ , where  $B$  is positive constant to be determined later. Then for any  $B > 0$  there exists a positive function  $\xi : M^n \times [0, T] \rightarrow \mathbb{R}$  such that

$$\left(\frac{\partial}{\partial t} - \Delta\right)\xi \geq B\xi, \quad (3.11)$$

$$|\nabla \xi| \leq D'\xi, \quad (3.12)$$

$$\xi(x, t) \geq e^{D'^{-1}(d_{g(t)}(O, x)+1)}, \quad (3.13)$$

on  $M^n \times [0, T]$ .

We claim that for all  $\epsilon > 0$  we have  $Q(x, y, t) > 0$  for all  $x \neq y$  and  $t \geq 0$ . Assuming the claim and taking the limit as  $\epsilon \rightarrow 0$ , we obtain Theorem 1.7. We prove the claim by contradiction. Notice that

$$\begin{aligned} \frac{Q}{d^2} &= \frac{H_x}{2} + \delta \frac{\langle X(y, t) - X(x, t), \nu_x \rangle}{d^2} + \epsilon \frac{\zeta}{d^2} \\ &\geq \frac{H_x}{2} + \delta \frac{\langle X(y, t) - X(x, t), \nu_x \rangle}{d^2} + \epsilon \frac{e^{D'^{-1}(d_{g(t)}(O, x)+1)} + e^{D'^{-1}(d_{g(t)}(O, y)+1)}}{d^2}. \end{aligned}$$

Since  $\frac{H_x}{2} + \delta \frac{\langle X(y, t) - X(x, t), \nu_x \rangle}{d^2}$  is uniformly bounded on  $[0, T]$ , for some  $K_1$  sufficiently large independent on  $t$  and  $B$ , we have  $Q > 0$  when  $d_{g(t)}(O, x) \geq K_1$  or  $d_{g(t)}(O, y) \geq K_1$ , and for some  $k_1$  sufficient small independent on  $t$  and  $B$ , we have  $Q > 0$  when  $d \leq k_1$ . Now suppose that the claim is false. Then there exists a first time  $t_0 > 0$ , the points  $x_0 \neq y_0$  such that  $Q(x_0, y_0, t_0) = 0$  and  $Q(x, y, t) > 0$  for all  $x, y \in M^n$  and  $t < t_0$ , moreover,

$$k_1 \leq d(x_0, y_0, t_0) \leq d_{g(t_0)}(O, x_0) + d_{g(t_0)}(O, y_0) \leq 2K_1, \quad (3.14)$$

where  $k_1$  and  $K_1$  are independent of  $B$ . Then at  $(x_0, y_0, t_0)$ , we have  $\frac{\partial Q}{\partial x_i} = 0$ ,  $\frac{\partial Q}{\partial y_i} = 0$ . It follows that at  $(x_0, y_0, t_0)$

$$-dH_x \langle \eta, \partial_i^x \rangle + \frac{d^2}{2} \nabla_i H_x + \delta dh_{iq}^x g_x^{qp} \langle \eta, \partial_p^x \rangle + \epsilon \frac{\partial \zeta}{\partial x^i} = 0, \quad (3.15)$$

and

$$dH_x \langle \eta, \partial_{y^i} \rangle + \delta \langle \partial_{y^i}, \nu_x \rangle + \epsilon \frac{\partial \zeta}{\partial y^i} = 0. \quad (3.16)$$

At  $(x_0, y_0, t_0)$  we have

$$\begin{aligned}
0 &\geq \frac{\partial Q}{\partial t} - \sum_{i,j=1}^n (g_x^{ij} \frac{\partial^2 Q}{\partial x^i \partial x^j} + 2g_x^{ik} g_y^{jl} \langle \partial_k^x, \partial_l^y \rangle \frac{\partial^2 Q}{\partial x^i \partial y^j} + g_y^{ij} \frac{\partial^2 Q}{\partial y^i \partial y^j}) \\
&= \epsilon \left( \frac{\partial}{\partial t} - \Delta_x \right) \xi + \epsilon \left( \frac{\partial}{\partial t} - \Delta_y \right) \xi - \epsilon (|A^x|^2 + \frac{4(H_x - \delta h_{nn}^x)}{d^2 H_x} \langle \partial_n^x, v_y \rangle^2) \zeta \\
&\quad + \epsilon \frac{4(H_x - \delta h_{nn}^x)}{d H_x} \frac{\partial \xi}{\partial y^q} \langle \eta, \partial_q^y \rangle \langle \partial_n^x, v_y \rangle^2 \\
&\quad - \epsilon \frac{4(H_x - \delta h_{nn}^x)}{d H_x} \frac{\partial \xi}{\partial y^q} \langle \eta, v_y \rangle \langle \partial_n^x, v_y \rangle \langle \partial_q^y, \partial_n^x \rangle. \tag{3.17}
\end{aligned}$$

Since the mean curvature is strictly positive on  $[0, t_0]$  and by (3.14), we have at  $(x_0, y_0, t_0)$

$$H_x \geq k_2 > 0, \tag{3.18}$$

where  $k_2$  is a constant independent of  $B$ . Notice that the second fundamental form is bounded by  $C_0$  on  $[0, T]$ . Combining with (3.11), (3.12), (3.14), (3.17) and (3.18), we get at  $(x_0, y_0, t_0)$

$$0 \geq \epsilon(B - C')\zeta, \tag{3.19}$$

where  $C'$  is a positive constant depending on  $C_0, C_m, k_1, k_2, K_1, D'$  and independent on  $B$ . Taking  $B = C' + 1$ , we obtain a contradiction.  $\square$

Recall Haslhofer and Kleiner [7] proved the following local estimate (See Theorem 1.8 in [7]).

**Theorem 3.2.** [7] *For any  $\delta > 0$  there exist  $\rho(\delta) > 0$  and  $C_l(\delta) < +\infty$  such that if  $M_t$  satisfies  $\delta$ -Andrews' noncollapsing condition in the parabolic ball  $P(x, t, r) = B(x, r) \times (t - r^2, t]$  with  $H(p, t) \leq r^{-1}$  then*

$$\sup_{P(x, t, \rho r)} |\nabla^l A| \leq C_l(\delta) r^{-(l+1)}. \tag{3.20}$$

As the corollary of Theorem 1.7 and Theorem 3.2, we have the following

**Corollary 3.3.** *Let  $M_t$  be the mean curvature flow for mean-convex complete noncompact embedded hypersurface in  $\mathbb{R}^{n+1}$  with bounded curvature at each time slice. If  $M_0$  satisfies the  $\delta$ -Andrews' noncollapsing condition, then  $\frac{|\nabla^l A|}{H^{l+1}} \leq C_l(\delta)$  for any  $t \in (0, T)$ .*

*Proof.* By Theorem 1.7,  $M_t$  satisfies the  $\delta$ -Andrews' noncollapsing condition for all  $t \geq 0$ . Then Corollary 3.3 follows from Theorem 3.2 directly.  $\square$

Also recall Haslhofer and Kleiner [7] proved (see Corollary 2.15 in [7])

**Theorem 3.4.** [7] *If  $M_t$  is an ancient mean-convex smooth mean curvature flow satisfies the  $\delta$ -Andrews' noncollapsing condition, then  $M_t$  is weakly convex.*

Finally as the application of Theorem 1.7 and Theorem 3.4, we give a proof of Theorem 1.8.

**Proof of Theorem 1.8.** By Theorem 1.7,  $M_t$  satisfies the  $\delta$ -Andrews' noncollapsing condition for all  $t \geq 0$ . By taking  $y \rightarrow x$  in the  $\delta$ -Andrews' noncollapsing condition, we have  $-Hg_{ij} \leq \delta h_{ij} \leq Hg_{ij}$  for all  $t \geq 0$ . Then the second fundamental forms are uniformly bounded for the recaled sequence (1.6). Hence the limit of the recaled sequence (1.6) is a mean-convex eternal solution  $M_\infty$  satisfying  $\delta$ -Andrews' noncollapsing condition. It follows from Theorem 3.4 that the  $M_\infty$  is weakly convex. Then Corollary 1.8 follows from the strong maximum principle and Hamilton's Harnack inequality (See Main Theorem B in [14]).  $\square$

#### 4. PROOF OF THEOREM 1.5 AND ITS APPLICATION

**Proof of Theorem 1.5.** Assume that  $M_t = x(M^n, t)$  be the Type III solution to the mean curvature flow with  $\sup_{M^n \times [0, \infty)} t|A|^2 = C < \infty$  and  $\tilde{x}(\cdot, s)$

be its corresponding normalized mean curvature flow (1.8). Denote  $\tilde{M}_s = \tilde{x}(M^n, s)$ . By (1.7), we have  $\sup_{M^n \times [0, \infty)} |\tilde{A}|^2 = C < \infty$ ,  $\tilde{W} = W$  and  $\tilde{H} = \sqrt{2t+1}H$ . Hence

$$\left(\frac{\partial}{\partial s} - \tilde{\Delta}\right)\tilde{W} = |\tilde{A}|^2\tilde{W},$$

and

$$\left(\frac{\partial}{\partial s} - \tilde{\Delta}\right)\tilde{H} = |\tilde{A}|^2\tilde{H} + \tilde{H}.$$

It follows from Corollary 3.3 that

$$\frac{|\tilde{\nabla}^l \tilde{A}|}{\tilde{H}^{l+1}} \leq C_l(\delta). \quad (4.1)$$

Hence we have

$$\frac{|\tilde{\nabla} \tilde{H}|}{\tilde{H}} \leq m_1, \quad (4.2)$$

and

$$\frac{|\frac{\partial}{\partial s} \tilde{H}|}{\tilde{H}} \leq m_2, \quad (4.3)$$

where  $m_1$  and  $m_2$  are positive constants depending on  $\delta$  and  $\sup_{M^n \times [0, \infty)} |\tilde{A}|$ . Then

$$\begin{aligned}
\left(\frac{\partial}{\partial s} - \tilde{\Delta}\right) \frac{\tilde{W}^2}{\tilde{H}^2} &= \frac{(\frac{\partial}{\partial s} - \tilde{\Delta})\tilde{W}^2}{\tilde{H}^2} - \frac{\tilde{W}^2(\frac{\partial}{\partial s} - \tilde{\Delta})|\tilde{H}|^2}{\tilde{H}^4} + 2\tilde{\nabla} \log \tilde{H}^2 \cdot \tilde{\nabla} \frac{\tilde{W}^2}{\tilde{H}^2} \\
&= -2\frac{\tilde{W}^2}{\tilde{H}^2} - \frac{2|\tilde{\nabla} \tilde{W}|^2}{\tilde{H}^2} + \frac{2\tilde{W}^2|\tilde{\nabla} \tilde{H}|^2}{\tilde{H}^4} + 4\frac{\tilde{\nabla} \tilde{H}}{\tilde{H}} \cdot \tilde{\nabla} \frac{\tilde{W}^2}{\tilde{H}^2} \\
&= -2\frac{\tilde{W}^2}{\tilde{H}^2} - \frac{2|\tilde{\nabla} \tilde{W}|^2}{\tilde{H}^2} - \frac{2\tilde{W}^2|\tilde{\nabla} \tilde{H}|^2}{\tilde{H}^4} + 4\frac{\tilde{\nabla} \tilde{H}}{\tilde{H}} \cdot \frac{\tilde{W} \tilde{\nabla} \tilde{W}}{\tilde{H}^2} \\
&\quad + 2\frac{\tilde{\nabla} \tilde{H}}{\tilde{H}} \cdot \tilde{\nabla} \frac{\tilde{W}^2}{\tilde{H}^2} \\
&\leq -2\frac{\tilde{W}^2}{\tilde{H}^2} + 2\frac{\tilde{\nabla} \tilde{H}}{\tilde{H}} \cdot \tilde{\nabla} \frac{\tilde{W}^2}{\tilde{H}^2}.
\end{aligned}$$

We follow an idea of Ecker and Huisken in [8]. Define  $\eta_a(\tilde{x}) = 1 + a|\tilde{x}|^2$ ,  $\rho(\tilde{x}, s) = \eta_a^{\epsilon-1} e^{\beta s}$ , where  $a$  is positive constant to be determined later. We calculate that

$$\left(\frac{\partial}{\partial s} - \tilde{\Delta}\right)\eta_a = -2a(|\tilde{x}|^2 + n),$$

and hence

$$\left(\frac{\partial}{\partial s} - \tilde{\Delta}\right)\rho \leq (\beta + 2(1 - \epsilon)(an + 1))\rho.$$

Moreover,  $|\tilde{\nabla} \eta_a|^2 \leq 4an\eta_a$  and  $|\tilde{\nabla} \rho| \leq 2a^{\frac{1}{2}}\rho$ . Multiplying by a testing  $\rho$  we compute

$$\begin{aligned}
\left(\frac{\partial}{\partial s} - \tilde{\Delta}\right) \frac{\tilde{W}^2}{\tilde{H}^2} \rho &= \rho \left(\frac{\partial}{\partial s} - \tilde{\Delta}\right) \frac{\tilde{W}^2}{\tilde{H}^2} + \frac{\tilde{W}^2}{\tilde{H}^2} \left(\frac{\partial}{\partial s} - \tilde{\Delta}\right) \rho - 2\langle \tilde{\nabla} \frac{\tilde{W}^2}{\tilde{H}^2}, \tilde{\nabla} \rho \rangle \\
&\leq (\beta + 2an - 2\epsilon) \frac{\tilde{W}^2}{\tilde{H}^2} \rho + 2\rho \langle \tilde{\nabla} \log \tilde{H}, \tilde{\nabla} \frac{\tilde{W}^2}{\tilde{H}^2} \rangle - 2\langle \tilde{\nabla} \frac{\tilde{W}^2}{\tilde{H}^2}, \tilde{\nabla} \rho \rangle \\
&= (\beta + 2an - 2\epsilon) \frac{\tilde{W}^2}{\tilde{H}^2} \rho + 2(\tilde{\nabla} \log \tilde{H} - \rho^{-1} \tilde{\nabla} \rho) \cdot \tilde{\nabla} \left(\frac{\tilde{W}^2}{\tilde{H}^2} \rho\right) \\
&\quad - 2\frac{\tilde{W}^2}{\tilde{H}^2} \langle \tilde{\nabla} \log \tilde{H}, \tilde{\nabla} \rho \rangle + 2\frac{\tilde{W}^2}{\tilde{H}^2} \rho^{-1} |\tilde{\nabla} \rho|^2 \\
&\leq (\beta + 2an + 4m_1 a^{\frac{1}{2}} + 8a - 2\epsilon) \frac{\tilde{W}^2}{\tilde{H}^2} \rho \\
&\quad + 2(\tilde{\nabla} \log \tilde{H} - \rho^{-1} \tilde{\nabla} \rho) \cdot \tilde{\nabla} \left(\frac{\tilde{W}^2}{\tilde{H}^2} \rho\right).
\end{aligned}$$

Taking  $a$  and  $\beta$  small enough such that  $\beta + 2an + 4m_1 a^{\frac{1}{2}} + 8a - 2\epsilon < 0$ , we have

$$\left(\frac{\partial}{\partial s} - \tilde{\Delta}\right) \frac{\tilde{W}^2}{\tilde{H}^2} \rho \leq 2(\tilde{\nabla} \log \tilde{H} - \rho^{-1} \tilde{\nabla} \rho) \cdot \tilde{\nabla} \left(\frac{\tilde{W}^2}{\tilde{H}^2} \rho\right), \quad (4.4)$$

with  $|\tilde{\nabla} \log \tilde{H} - \rho^{-1} \tilde{\nabla} \rho| \leq m_1 + 2a^{\frac{1}{2}}$ .

$$\begin{aligned} \frac{\partial}{\partial s} \left( \frac{\tilde{W}^2}{\tilde{H}^2} \rho \right) &= 2 \frac{\langle \tilde{\nabla} \tilde{H}, \omega \rangle \tilde{W}}{\tilde{H}^2} \rho - 2 \frac{\frac{\partial}{\partial s} \tilde{H}}{\tilde{H}} \left( \frac{\tilde{W}^2}{\tilde{H}^2} \rho \right) + (\epsilon - 1) \frac{\frac{\partial}{\partial s} \eta_a}{\eta_a} \left( \frac{\tilde{W}^2}{\tilde{H}^2} \rho \right) + \beta \frac{\tilde{W}^2}{\tilde{H}^2} \rho \\ &\leq 2C_1(\delta) e^{\beta s} + C' \frac{\tilde{W}^2}{\tilde{H}^2} \rho, \end{aligned}$$

where  $C'$  is positive constant depends on  $\epsilon, \beta, m_2, a$  and  $\sup_{M^n \times [0, \infty)} |\tilde{A}|$ . So we get that  $\sup_{\tilde{M}_s} \frac{\tilde{W}^2}{\tilde{H}^2} \rho$  is finite at each time slice. Applying the maximum principle to (4.4) (see Corollary 1.1 in [8]) we have

$$\sup_{\tilde{M}_s} \frac{\tilde{W}^2}{\tilde{H}^2} (1 + a|\tilde{x}|^2)^{\epsilon-1} \leq e^{-\beta s} \sup_{\tilde{M}_0} \frac{\tilde{W}^2}{\tilde{H}^2} (1 + a|\tilde{x}|^2)^{\epsilon-1}. \quad (4.5)$$

Moreover,  $\tilde{H}$  is uniformly bounded by the Type III condition, hence  $\tilde{W} \rightarrow 0$  as  $s \rightarrow +\infty$  on any compact set. Now we can use the same arguments in the proof of Theorem 1.2 to get a contradiction.  $\square$

As an application to Theorem 1.5, we have following corollary.

**Corollary 4.1.** *Let  $(y, u(|y|))$  be the smooth convex entire graph over  $\mathbb{R}^n$ ,  $n \geq 2$ , satisfying  $\delta$ -Andrews' noncollapsing condition. Suppose that there exist positive constants  $\epsilon_0, c$  and  $N$  such that for  $r \geq N$*

$$u'(r) \geq cr^{\epsilon_0}. \quad (4.6)$$

*Then the long-time solution to mean curvature flow with initial data  $(y, u(|y|))$  must be Type IIb. Moreover, we have the long-time solution to mean curvature flow for graph  $(y, |y|^\alpha)$  over  $\mathbb{R}^n$ ,  $\alpha > 1$  and  $n \geq 2$ , must be Type IIb.*

*Proof.* We choose  $\omega = -e_{n+1}$ . Define  $r = |y|$ . Since  $(y, u(|y|))$  is convex,  $u''(r) > 0$  and  $u'(r) > 0$ . By (2.10) and (2.11), taking  $\epsilon = \frac{\epsilon_0}{2}$ , we have

$$\begin{aligned} \frac{H}{W} (1 + |x|^2)^{\frac{1-\epsilon}{2}} &\geq \frac{(n-1)u'(r)}{r} (1 + r^2)^{\frac{1-\epsilon}{2}} \\ &\geq c(n-1)r^{\frac{\epsilon_0}{2}}. \end{aligned}$$

for  $r \geq N$ . If the graph  $(y, u(|y|))$  is smooth and convex, then  $H$  and  $W$  are positive and continuous. Then (1.13) is satisfied for  $r \leq N$ . Hence the mean curvature flow for the graph  $(y, u(|y|))$  over  $\mathbb{R}^n$ ,  $n \geq 2$ , satisfying (4.6) must be Type IIb by Theorem 1.5. For the case the entire graph is  $(y, |y|^\alpha)$  over  $\mathbb{R}^n$ ,  $n \geq 2$  and  $\alpha > 1$ , it is straightforward to check the conditions of Theorem 1.5 are satisfied.  $\square$



## 5. PROOF OF TABLE 1

First we prove the following

**Theorem 5.1.** *Let  $M_t$  be the solution to mean curvature flow with initial data  $M_0$  is the graph  $(y, |y|^\alpha)$  over  $\mathbb{R}^n$ ,  $\alpha > 1$  and  $n \geq 2$ . If  $\alpha > 2$ , then the  $|A|(p, t) \rightarrow +\infty$  as  $t \rightarrow +\infty$  for any  $p \in M^n$ . If  $1 < \alpha < 2$ , then  $|A|(p, t) \rightarrow 0$  as  $t \rightarrow +\infty$  for any  $p \in M$ . If  $\alpha = 2$ , then  $0 < c_p \leq |A|(p, t) \leq 2n^2$ , where  $c_p$  is positive constant depending on  $p$  and independent on  $t$ .*

*Proof.* In [6] Altschuler, Steven J., L. F. Wu proved that there exists rotational symmetric convex translating graph  $X_0 = (y, g_N(|y|))$  satisfying

$$\vec{H} = V^\perp.$$

with  $V = Ne_{n+1}$  and  $\lim_{r \rightarrow +\infty} \frac{g'_N(r)}{r} = N$ , and hence it is "asymptotic to" paraboloid as  $r \rightarrow +\infty$ . The solution to mean curvature flow with initial data  $X_0$  is translating as  $X_t(\phi_t^*(p), t) = X_0(p) + Ne_{n+1}t$ .

If  $\alpha > 2$ , then for any large  $N$  there exists positive constant  $C_N$  such that  $|y|^\alpha + C_N > g_N(|y|)$ . It implies that  $M_0 + C_N e_{n+1}$  is above  $X_0$ . Now we argue by contradiction. Assume that there exists  $p \in M^n$  such that  $|A|(p, t) \leq C_p$ . By (1.1),

$$|(x(p, t) + C_N e_{n+1}) - (x(p, 0) + C_N e_{n+1})| \leq C_p t.$$

That implies

$$B(x(p, 0) + C_N e_{n+1}, C_p t) \cap (M_t + C_N e_{n+1}) \neq \emptyset.$$

However, taking  $N = 2C_p$ ,  $X_t$  is translating in the  $e_{n+1}$  direction with velocity  $2C_p$  which implies that  $B(x(p, 0) + C_N e_{n+1}, C_p t)$  will stand below  $X_t$  for  $t$  sufficient large. But  $M_t + C_N e_{n+1}$  is above  $X_t$  for all  $t \geq 0$  and hence

$$B(x(p, 0) + C_N e_{n+1}, C_p t) \cap M_t = \emptyset$$

for  $t$  sufficient large. Then we obtain a contradiction.

If  $\alpha = 2$ , by (2.10) and (2.11), a direct calculation shows that  $0 < H \leq 2nW$  at  $t = 0$ . It follows from Lemma 2.1 that  $0 < H \leq 2nW$  for  $t \geq 0$ . By the convexity,  $|A| \leq nH \leq 2n^2W \leq 2n^2$ . We use the contradictory arguments to get the lower bound for  $|A|(p, t)$ . Assume that there exists  $p \in M^n$  such that  $|A|(p, t) \rightarrow 0$ . By (1.1), we have for any small  $\epsilon > 0$

$$|x(p, t) - x(p, 0)| \leq \epsilon t. \quad (5.1)$$

Taking  $N = 1$ , there exists positive constant  $C_1$  such that  $|y|^2 + C_1 > g_1(|y|)$ . Hence  $M_0 + C_1 e_{n+1}$  is above  $X_0$ . Notice that  $X_t$  is translating in the  $e_{n+1}$  direction with velocity 1 which implies that  $B(x(p, 0) + C_1 e_{n+1}, \epsilon t)$  will stand below  $X_t$  for  $\epsilon < 1$  and  $t$  sufficient large. But  $M_t + C_1 e_{n+1}$  is above  $X_t$  for all  $t \geq 0$  and hence

$$B(x(p, 0) + C_1 e_{n+1}, \epsilon t) \cap (M_t + C_1 e_{n+1}) = \emptyset$$

for  $\epsilon < 1$  and  $t$  sufficient large, which contradicts to (5.1).

Finally we consider the case  $1 < \alpha < 2$ . Since  $M_t$  is rotational symmetric and convex for any  $t \geq 0$ , for any  $t$  there exists a point  $q_t \in M_t$  achieves the unique minimum of the graph function for  $M_t$ . Moreover,  $q_t$  always stays on the  $x_{n+1}$ -axis, otherwise by the symmetry there would have more than one minimum point on  $M_t$ . Denote  $x(p_0, 0) = q_0$  for some  $p \in M^n$ . Since the all unit normal vectors at  $q_t$  are  $-e_{n+1}$ , we have  $x(p_0, t) = q_t$  by (1.1) for all  $t \geq 0$ . Hence  $v(p_0, t) = -e_{n+1}$  for any  $t \geq 0$ . If  $H(p_0, t) \geq c > 0$ , then by (1.1)

$$|x(p_0, t) - x(p_0, 0)| \geq ct.$$

Since for any small  $\epsilon > 0$ , there exists positive constant  $C_N$  such that  $g_\epsilon(|y|) + C_N \geq |y|^\alpha$ . Hence for any small  $\epsilon > 0$  we have  $X_0 + C_N e_{n+1}$  is above  $M_0$ . Note that  $X_t$  is translating in the  $e_{n+1}$  direction with velocity  $\epsilon$  and  $M_t$  is always below  $X_t + C_N e_{n+1}$  for all  $t \geq 0$  and hence

$$|x(p_0, t) - x(p_0, 0)| \leq \epsilon t + C_N.$$

Then we obtain a contradiction when  $\epsilon < c$  and  $t$  is sufficient large. Hence  $H(p_0, t) \rightarrow 0$  as  $t \rightarrow +\infty$ . It follows from  $\frac{|\nabla H|}{H^2} \leq C_1(\delta)$  and the convexity that  $H(p, t) \rightarrow 0$  as  $t \rightarrow +\infty$  for any  $p \in M^n$ . By the convexity, we conclude that  $|A(p, t)| \rightarrow 0$  as  $t \rightarrow +\infty$  for any  $p \in M^n$ .  $\square$

Finally, we give the proof of Table 1. **Proof of Table 1.** Since the graph  $(y, |y|^\alpha)$  over  $\mathbb{R}^n$  for  $0 < \alpha \leq 1$  (smooth the graph at a neighborhood of the origin) satisfying (1.9) and (1.10), the mean curvature flow for such graph is Type III and the normalized mean curvature flow (1.8) converges as  $s \rightarrow \infty$  to a self-expander by Ecker and Huisken's results (Corollary 4.4 in [8]). The rest of Table 1 follows from Corollary 4.1 and Theorem 5.1.  $\square$

## 6. APPENDIX

In this section we prove that the limit of rescaled sequence (1.7) for convex Type III mean curvature flow is self-expander (Corollary 6.3). Similar results had been obtained by Hamilton [15] [14] for Type II Ricci flow and mean curvature flow and Chen and Zhu [3] for Type III Ricci flow. One can use the similar arguments to prove Corollary 6.3. We give a proof for sake of convenience for the readers.

First we recall Hamilton's Harnack inequality for the mean curvature flow.

**Theorem 6.1.** [14] *For any weak convex solution to mean curvature flow for  $t > 0$  we have*

$$\tilde{Z} = \frac{\partial H}{\partial t} + \frac{H}{2t} + 2V_i \nabla_i H + h_{ij} V_i V_j \geq 0, \quad (6.1)$$

for all tangent vectors  $V$ .

**Theorem 6.2.** *Any strictly convex solution to the mean curvature flow where  $\frac{\partial}{\partial t}(\sqrt{t}H) = 0$  at some point  $(x_0, t_0)$  for  $t_0 > 0$  must be the self-expander.*

*Proof.* Recall Hamilton proved (see Corollary 4.4 in [14]) the Harnack quantity  $\tilde{Z}$  satisfies

$$(D_t - \Delta)\tilde{Z} = (|A|^2 - \frac{2}{t})\tilde{Z} + 2\tilde{X}_a\tilde{U}_a - 2h_{bc}\tilde{Y}_{ab}\tilde{Y}_{ac} - 4\tilde{Y}_{ab}\tilde{W}_{ab}, \quad (6.2)$$

with  $\tilde{X}_a = \nabla_a H + h_{ab}V_b$ ,  $\tilde{Y}_{ab} = \nabla_a V_b - Hh_{ab} - \frac{1}{2t}g_{ab}$ ,  $\tilde{W}_{ab} = \frac{\partial}{\partial t}h_{ab} + V_c\nabla_ch_{ab} + \frac{1}{2t}h_{ab}$ ,  $\tilde{U}_a = (\frac{\partial}{\partial t} - \Delta)V_a + h_{ab}\nabla_b H + \frac{1}{t}V_a$ . Since  $\frac{\partial}{\partial t}(\sqrt{t}H) = 0$  at some point  $(x_0, t_0)$  for  $t_0 > 0$ , we know that at this point

$$\frac{\partial H}{\partial t} + \frac{H}{2t} = 0. \quad (6.3)$$

Taking  $V_i = -h_{ij}^{-1}\nabla_j H$  in (6.1), we have at  $(x_0, t_0)$

$$-h_{ij}^{-1}\nabla_i H \nabla_j H \geq 0.$$

It follows that at  $(x_0, t_0)$

$$\nabla H = 0. \quad (6.4)$$

Then we obtain that  $\tilde{Z} = 0$  in the  $V = 0$  direction. The strong maximum principle implies that there exists vector  $V$  at each point such that  $\tilde{Z} = 0$ . Moreover, the zero factor  $V$  is obtained from the first variation of  $\tilde{Z}$  by  $V_a = -h_{ab}^{-1}\nabla_b H$ .

Now fix  $V_a = -h_{ab}^{-1}\nabla_b H$  at  $(x_0, t_0)$  and extend  $V$  in a neighborhood of  $(x_0, t_0)$  in space-time such that

$$\tilde{U}_a = (\frac{\partial}{\partial t} - \Delta)V_a + h_{ab}\nabla_b H + \frac{1}{t}V_a = \tilde{X}_a,$$

and

$$\tilde{Y}_{ab} = \nabla_a V_b - Hh_{ab} - \frac{1}{2t}g_{ab} = -\tilde{W}_{ad}h_{db}^{-1}.$$

Then at  $(x_0, t_0)$

$$\begin{aligned} 0 &\geq (\frac{\partial}{\partial t} - \Delta)\tilde{Z} \\ &= (|A|^2 - \frac{2}{t})\tilde{Z} - 4\tilde{W}_{ab}\tilde{Y}_{ab} + 2\tilde{X}_a\tilde{U}_a - 2h_{ac}\tilde{Y}_{bc}\tilde{Y}_{ba} \\ &= 4\tilde{W}_{ad}h_{db}^{-1}\tilde{W}_{ba} + 2|\tilde{X}_a|^2 - 2h_{ac}\tilde{W}_{bd}h_{dc}^{-1}\tilde{W}_{be}h_{ea}^{-1} \\ &= 2\tilde{W}_{ad}h_{db}^{-1}\tilde{W}_{ba} + 2|\tilde{X}_a|^2, \end{aligned}$$

which implies that

$$\tilde{W}_{ab} = 0, \quad \tilde{X}_a = 0.$$

Thus we obtain

$$\frac{\partial}{\partial t} h_{ab} + V_c \nabla_c h_{ab} + \frac{1}{2t} h_{ab} = 0, \quad \nabla_a H + h_{ab} V_b = 0, \quad (6.5)$$

for  $V_a = -h_{ab}^{-1} \nabla_b H$  everywhere. We get from differentiating the second equation in (6.5) that

$$\nabla_a \nabla_b H + V_c \nabla_a h_{bc} + h_{bc} \nabla_a V_c = 0. \quad (6.6)$$

It follows from Theorem 2.3 in [15] that

$$\begin{aligned} \nabla_a \nabla_b H &= \nabla_a \nabla_c h_{bc} \\ &= \nabla_c \nabla_a h_{bc} + R_{acbd} h_{dc} + R_{accd} h_{bd} \\ &= \Delta h_{ab} + (h_{ab} h_{cd} - h_{ad} h_{bc}) h_{dc} + (h_{ac} h_{cd} - h_{ad} H) h_{bd} \\ &= \frac{\partial}{\partial t} h_{ab} - H h_{ad} h_{bd}. \end{aligned} \quad (6.7)$$

By (6.5), (6.6) and (6.7), we get

$$\nabla_a V_c = H h_{ac} + \frac{1}{2t} g_{ac}. \quad (6.8)$$

Consider the vector

$$\frac{1}{2t} T^\alpha = g^{ij} V_i \nabla_j X^\alpha + H v^\alpha + \frac{1}{2t} X^\alpha,$$

where  $v = (v^1, \dots, v^{n+1})$  is the unit normal vector of  $X$ . By (6.5) and (6.8), we have

$$\nabla_k T^\alpha = g^{ij} \nabla_k V_i \nabla_j X^\alpha + g^{ij} V_i h_{jk} v^\alpha + (\nabla_k H) v^\alpha - h_{kj} g^{jm} \nabla_m X^\alpha + \frac{1}{2t} \nabla_k X^\alpha = 0.$$

Then

$$g^{ij} V_i \nabla_j X^\alpha + H v^\alpha + \frac{1}{2t} (X^\alpha - T^\alpha) = 0.$$

It follows that  $T$  is a constant vector. Taking the vertical part, we have

$$H v^\alpha + \frac{1}{2t} (X^\alpha - T^\alpha)^\perp = 0.$$

□

Finally, we give the proof of Corollary 6.3.

**Corollary 6.3.** *Let  $M_t$  be the Type III convex mean curvature flow for the noncompact hypersurface with bounded second fundamental form at each time slice. Then the limit obtained as (1.7) is a non-flat self-expander splitting as  $\mathbb{R}^{n-k} \times \Sigma^k$ , where  $\Sigma^k$  is strictly convex.*

*Remark 6.4.* Due to a counter-example in [5](see Example 3.4 in [5]), Corollary 6.3 is not true if we only assume the Type III mean curvature flow is mean-convex.

*Proof.* By Hamilton's Harnack (6.1), we have  $\sqrt{t}H$  is pointwisely monotone nonincreasing. Taking  $\tilde{x}_i(s) = \tilde{x}(s + s_i)$ , where  $\tilde{x}$  is defined in (1.7) and  $s_i \rightarrow +\infty$ . We take the limit as the way in the proof of Theorem 1.2. Let  $p \in M^n$  be the based point taken in the proof of Theorem 1.2. Then  $\tilde{H}_\infty(p_\infty, s) = \lim_{i \rightarrow \infty} \tilde{H}(p, s + s_i) = \lim_{i \rightarrow \infty} \sqrt{2(t + t_i) + 1}H(p, t + t_i) = \sqrt{2} \lim_{i \rightarrow \infty} \sqrt{t + t_i}H(p, t + t_i) \equiv \text{constant} > 0$ , where  $s_i = \frac{1}{2} \log(2t_i + 1)$ . Then strong maximum principle we know the limit splitting as  $\mathbb{R}^{n-k} \times \Sigma^k$ , where  $\Sigma^k$  is strictly convex. Hence Corollary 6.3 holds by Theorem 6.2.  $\square$

## REFERENCES

- [1] B.Andrews. *Non-collapsing in mean-convex mean curvature flow*. Geometry & Topology, 16(3), 1413-1418, 2012.
- [2] Chow, Bennett, et al. *The Ricci Flow: Techniques and Applications: Part II: Analytic Aspects*. Mathematical Surveys & Monographs, 144 American Math Soc American Mathematical Society, 2007:536.
- [3] Chen, Bing Long, and X. P. Zhu. *Complete Riemannian manifolds with pointwise pinched curvature*. Inventiones Mathematicae 140.2(2000):423-452.
- [4] Chen, Jingyi, and W. He. *A note on singular time of mean curvature flow*. Mathematische Zeitschrift 266.4(2010):921-931.
- [5] Cheng Liang, Sesum Natasa, *Asymptotic behavior of Type III mean curvature flow*, <http://arxiv.org/abs/1403.0235v3>
- [6] Altschuler, Steven J., L. F. Wu. *Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle*. Calculus of Variations & Partial Differential Equations 2.1(1994):101-111.
- [7] Haslhofer R, Kleiner B. *Mean Curvature Flow of Mean Convex Hypersurfaces*. Communications on Pure & Applied Mathematics, 2016, online.
- [8] K.Echer, G.Huisken, *mean curvature evolution of entire graphs*. Ann. Math. 130 (1989). 453-471
- [9] K.Ecker, G.Huisken, *Interior estimates for hypersurfaces moving by mean curvature*. Invent. Math. 105 (1991), 547-569
- [10] G.Huisken, *Asymptotic behavior for singularities of the mean curvature flow*. J. Differential Geom. 31 (1990), no. 1, 285-299.
- [11] G. Huisken and C. Sinestrari, *Mean curvature flow singularities for mean convex surface*, Calc. Var. PDE, 8(1999), 1-14.
- [12] G. Huisken and C. Sinestrari, *Convexity estimates for mean curvature flow and singularities of mean convex surfaces*, Acta Math., 183(1999). 47-70.
- [13] Huisken G, Sinestrari C. *Mean curvature flow with surgeries of two-convex hypersurfaces*. Inventiones Mathematicae, 2009, 175(1):137-221.
- [14] R.Hamilton, *Harnack estimate for the mean curvature flow*, J.Differential. Geom., 41 (1995) , 215-226.
- [15] R.Hamilton, *Formation of singularities in the Ricci flow*. Surveys in Diff. Geom. 2 (1995), 7-136.
- [16] Tom Ilmanen, *Singularities of mean curvature flow of surfaces*, preliminary version, available under <http://www.math.ethz.ch/~ilmanen/papers/sing.ps>.
- [17] Weimin Sheng and Xu-Jia Wang, *Singularity profile in the mean curvature flow*, Methods Appl. Anal. 16 (2009), no. 2, 139-155.

SCHOOL OF MATHEMATICS AND STATISTICS & HUBEI KEY LABORATORY OF MATHEMATICAL SCI-  
ENCES, CENTRAL CHINA NORMAL UNIVERSITY, WUHAN, 430079, P.R.CHINA  
*E-mail address:* chengliang@mail.ccnu.edu.cn